International Journal of Theoretical Physics, Vol. 33, No. 4, 1994

Dislocations and Internal Length Measurement in Continuized Crystals. II. Closed Teleparallelism

Andrzej Trzęsowski[1]

Received September 20, 1993

The Burgers field responsible for dislocations in a continuized crystal is represented by the torsion tensor of a teleparallel connection, metric with respect to the internal length measurement metric tensor. Lattice lines in a continuized dislocated monocrystal are represented by geodesics of the teleparallel connection, and the internal length measurement along these geodesics is analyzed. The closed teleparallelism responsible for uniformly dense distributions of dislocations is discussed, and equations describing slip surfaces for such distributions of dislocations are formulated. The Galilei-like character of the geometry describing uniformly dense distributions of dislocations is pointed out.

1. INTRODUCTION

The existence of many dislocations breaks the long-range order of a crystalline solid in a special manner, manifesting itself in the existence of different short-range orders in macroscopically small neighborhoods of different points of the body. In a continuous limit, called the *continuized crystal* (Kröner, 1984, 1986; Trzęsowski, 1993), the distribution of these short-range orders is defined by a triple $(\Phi, G, \mathbf{g})$, where $\Phi = (\mathbf{E}_a)$ is a moving frame globally defined on the body $\mathscr{B}$ identified with its distinguished reference configuration $\mathscr{B}_\kappa \subset E^3$, an open and connected subset of the three-dimensional Euclidean point space E^3 (Trzęsowski, 1993), $G \subset SO(3)$ is a group of rotations describing material symmetries of the considered macroscopically homogeneous crystalline solid (Trzęsowski, 1994, Section 2), and $\mathbf{g}$ is a metric tensor with respect to which the moving

[1]Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland.

0020-7748/94/0400-0951$07.00/0

frame Φ is orthonormal, i.e.,

$$(\mathbf{E}_a, \mathbf{E}_b)_g = \delta_{ab} \tag{1}$$

where $(\cdot, \cdot)_g$ denotes the scalar product on the material Riemannian space $(\mathscr{B}, \mathbf{g})$. This metric tensor represents the property of the dislocated crystalline solid that dislocations have no influence on the local metric properties of a crystal structure of the body (Kröner, 1985) and, at the same time, this metric tensor is responsible for point defects created by these dislocations (Trzęsowski, 1994). The moving frame Φ defines a system of the three indendent congruences of curves—trajectories of the vector fields $\mathbf{E}_a$, $a = 1, 2, 3$. The tangents of these trajectories constitute at each point of the body a triad of the so-called *local crystallographic directions*. The material symmetry group G defines, at each point of the body, the rotational equivalence of the triads. This local rotational uncertainty in selecting the triad represents the existence of local material symmetries in the continuized dislocated crystal. Moreover, the base vector fields $\mathbf{E}_a$ define locally scales of an *internal length measurement* along (local) crystallographic directions. Consequently, for a continuized dislocated monocrystal, trajectories of the vector fields $\mathbf{E}_a$ can be interpreted as *lattice lines* in this crystal (Bilby *et al.*, 1958). However, these vector fields cannot be interpreted (even locally and in a dislocated monocrystal), in contradiction to base vectors of a Bravais lattice, as those defining translational material symmetries of the continuized crystal. This is because in a continuized crystal translational symmetries of a discrete crystal structure are lost, and only its rotational symmetries (represented by the group G) are preserved (Trzęsowski, 1993).

The long-range distortion of a crystalline solid material structure due to dislocations can be described by means of the so-called *Burgers field* $\tau_\Phi = (\tau^a)$ defined by

$$\tau^a = dE^a = \tfrac{1}{2}\tau^a{}_{bc} E^b \wedge E^c \tag{2a}$$

$$\tau^a{}_{bc} = -C^a_{bc}, \qquad \langle E^a, \mathbf{E}_b \rangle = \delta^a_b \tag{2b}$$

where $\Phi^* = (E^a)$ is the moving coframe dual to Φ, and

$$[\mathbf{E}_a, \mathbf{E}_b] = C^c_{ab} \mathbf{E}_c \tag{3}$$

where $[\mathbf{u}, \mathbf{v}] = \mathbf{u} \circ \mathbf{v} - \mathbf{v} \circ \mathbf{u}$ is the commutator product (bracket) of vector fields $\mathbf{u}$ and $\mathbf{v}$ considered as first-order differential operators (Trzęsowski, 1993). The Hodge dual fields α^a of the 2-forms τ^a, i.e.,

$$\begin{aligned} \alpha^a &= *\tau^a = \alpha^{ba} E_b \\ E_a &= \delta_{ab} E^b \end{aligned} \tag{4}$$

where $*$ denotes the Hodge star operator on $(\mathscr{B}, \mathbf{g})$ (Von Westenholz, 1978), univocally define the dislocation density tensor $\boldsymbol{\alpha}$ of the form (Trzęsowski, 1994)

$$\boldsymbol{\alpha} = \alpha^{ab}\mathbf{E}_a \otimes \mathbf{E}_b$$
$$\alpha^{ba} = \tfrac{1}{2}\tau^{a}{}_{cd}e^{cdb} \tag{5}$$

where $e^{abc} \underset{*}{=} \epsilon^{abc}$ denotes (in the base Φ) the permutation symbol. Equivalently,

$$\tau^{a}{}_{bc} = \alpha^{pa}e_{pbc} = e_{bcd}\gamma^{da} - t_{[b}\delta^{a}_{c]} \tag{6}$$

where

$$\gamma^{ab} = \alpha^{(ab)}$$
$$t_a = \tau^{b}{}_{ba} = e_{abc}\alpha^{bc} \tag{7}$$

On the other hand, if the considered body is a three-dimensional connected manifold, there exists a one-to-one correspondence between globally defined moving frames $\Phi = (\mathbf{E}_a)$ and the so-called teleparallel covariant derivatives $\nabla^{\Phi} = (\Gamma^{A}_{BC}[\Phi])$. This correspondence is defined by the condition of covariant constancy of vector fields $\mathbf{E}_a$, $a = 1, 2, 3$:

$$\nabla^{\Phi}\mathbf{E}_a = 0 \tag{8}$$

and by the demand that the curvature tensor of ∇^{Φ} vanishes. The local metric properties of a continuized dislocated crystal [see the commentary following equation (1)] are then represented by the condition of ∇^{Φ}-covariant constancy of the internal length measurement metric tensor:

$$\nabla^{\Phi}\mathbf{g} = 0 \tag{9}$$

and it is a property of $\mathbf{g}$ invariant under its global rescaling defined by

$$\Phi \to \Phi\mathbf{L} = (\mathbf{E}_a L^{a}{}_{b})$$
$$\mathbf{L} = \| L^{a}{}_{b} \| \in GL^{+}(3) \tag{10}$$

where $GL^{+}(3)$ denotes the group of all real 3×3 matrices with positive determinant. Since for

$$\mathbf{E}_a = \underset{a}{e}{}^{A}\partial_A, \qquad E^a = \overset{a}{e}_A\, dX^A$$
$$\underset{a}{e}{}^{A}\overset{b}{e}_A = \delta^{b}_{a} \tag{11}$$

we have

$$\Gamma^{C}_{AB}[\Phi] = \underset{a}{e}{}^{C}\partial_B\overset{a}{e}_A \tag{12}$$

Therefore, the torsion tensor τ_Φ of the teleparallel connection has the form

$$\tau_\Phi = \mathbf{E}_a \otimes \tau^a = S^a{}_{bc}\, \mathbf{E}_a \otimes E^b \otimes E^c$$
$$S^a{}_{bc} = \tfrac{1}{2}\tau^a{}_{bc} = \overset{a}{e}_A \underset{b}{e}^B \underset{c}{e}^C \Gamma^A_{[BC]}[\Phi] \tag{13}$$

and τ_Φ is invariant under the rescaling (10): $\tau_{\Phi L} = \tau_\Phi$. Thus, the tensor τ_Φ can be interpreted as a representation of the dislocation density tensor invariant with respect to the global rescaling of the internal length measurement metric tensor.

It follows from the condition (8) that trajectories of the vector fields $\mathbf{E}_a$ are ∇^Φ-geodesics. Moreover, every ∇^Φ-geodesic (henceforth called a Φ-geodesic) is an integral curve of a certain ∇^Φ-parallel (henceforth called Φ-parallel) vector field, i.e., a field $\mathbf{v}$ such that

$$\nabla^\Phi \mathbf{v} = 0 \tag{14a}$$

i.e.,

$$\mathbf{v} = v^a \mathbf{E}_a, \qquad v^a = \text{const} \tag{14b}$$

We see that lattice lines in a continuized dislocated monocrystal can be considered as Φ-geodesics located in the Riemannian material space $(\mathscr{B}, \mathbf{g})$. Consequently, an internal length measurement along these lattice lines can be considered in terms of differences between Φ-geodesics and $\mathbf{g}$-geodesics. In the paper, these differences are described in terms of the dislocation density tensor $\boldsymbol{\alpha}$ (Section 2). Moreover, transformations preserving the internal length measurement along lattice lines are discussed (Section 2), and uniformly dense distributions of dislocations (Trzęsowski, 1987) are considered (Sections 3 and 4).

2. INTERNAL LENGTH MEASUREMENT ALONG LATTICE LINES

It is known that Φ-geodesics are uniquely determined by the symmetric part $\overset{\circ}{\nabla} = (\overset{\circ}{\Gamma}{}^A_{BC})$ of the teleparallel covariant derivative $\nabla^\Phi = (\Gamma^A_{BC}[\Phi])$ defined by

$$\overset{\circ}{\nabla} = \nabla^\Phi - \tau_\Phi$$
$$\overset{\circ}{\Gamma}{}^A_{BC} = \Gamma^A_{BC}[\Phi] - S^A_{BC} \tag{15}$$

where $\Gamma^A_{BC}[\Phi]$ is given by equation (12), and [see equations (11) and (13)]

$$\tau_\Phi = \partial_A \otimes \tau^A = S^A{}_{BC}\, \partial_A \otimes dX^B \otimes dX^C$$
$$\tau^A = \underset{a}{e}^A \tau^a = \tfrac{1}{2}\tau^A{}_{BC}\, dX^B \wedge dX^C \tag{16}$$
$$S^A{}_{BC} = \tfrac{1}{2}\tau^A{}_{BC} = \Gamma^A_{[BC]}[\Phi], \qquad [\tau^A{}_{BC}] = [l^{-1}]$$

where $[l] = \text{cm}$ in the cgs units system, and the so-called geometric frame reference $X = (X^A)$ ($[X^A] = [dX^A] = [l]$, $[\partial_A] = [l^{-1}]$) has been used. Let $X^A(\tau)$ be a $\overset{\circ}{\nabla}$-geodesic. Denote by $v^A = dX^A/d\tau$ the tangent to $X^A(\tau)$. If τ is an affine parameter on the geodesic, then v^A satisfies the geodesic equation (Schouten, 1954):

$$\frac{dv^A}{d\tau} + \overset{\circ}{\Gamma}{}^A_{BC} v^B v^C = 0$$
$$[v^A] = [1], \qquad [\tau] = [d\tau] = [l] \tag{17}$$

The affine geodesic parameter τ offers a means of defining intervals along a geodesic that is independent of a metric. On the other hand, the internal length measurement metric tensor **g** (Section 1) defines the proper interval for all curves on the manifold $\mathscr{B}$. It seems natural, therefore, to require that the two be in agreement along $\overset{\circ}{\nabla}$-geodesics. If we denote the **g**-metric interval by s, then [see equations (1) and (11)]

$$ds = (g_{AB} v^A v^B)^{1/2}\, d\tau \tag{18a}$$

$$g_{AB} = \overset{a}{e}_A \overset{b}{e}_B \delta_{ab}, \qquad [s] = [l] \tag{18b}$$

and the condition for agreement between two intervals is (in differential form) (Bradfield, 1990)

$$\frac{d^2 s}{d\tau^2} = 0 \tag{19}$$

which is equivalent to

$$s = a\tau + b$$
$$[a] = [1], \qquad [b] = [l] \tag{20}$$

where a, b are constants.

Let us denote

$$Q_{CAB} = \overset{\circ}{\nabla}_C g_{AB} \tag{21}$$

Then [see equations (12), (15), (16), and (18b)]

$$Q_{CAB} = \tau_{(AB)C}, \qquad \tau_{ABC} = g_{AD} \tau^D{}_{BC}$$
$$\tau_{ABC} = e_{BCD} \gamma^D{}_A - t_{[B} g_{C]A} = -\tau_{ACB}, \qquad \gamma^D{}_B = g_{BC} \gamma^{DC} \tag{22}$$

where we have denoted [see equations (5)–(7)]

$$\alpha^{BA} = \tfrac{1}{2} \tau^A{}_{CD} e^{CDB}, \qquad \tau^A{}_{BC} = e_{BCD} \alpha^{DA} \tag{23a}$$

$$\gamma^{AB} = \alpha^{(AB)}, \qquad \alpha^{[AB]} = \tfrac{1}{2} t_C e^{CAB} \tag{23b}$$

$$t_A = \tau^B{}_{BA} = e_{ABC} \alpha^{BC} \tag{23c}$$

and

$$e^{ABC} = g^{-1/2}\epsilon^{ABC}, \qquad e_{ABC} = g^{1/2}\epsilon_{ABC}$$
$$g = \det\|g_{AB}\| = e_{\Phi}^{-2}, \qquad e_{\Phi} = \det\|\underset{a}{e}^{A}\| > 0 \tag{24}$$

where $\epsilon^{ABC} = \epsilon_{ABC}$ denotes the permutation symbol. It follows from equations (21) and (22) that the connection coefficients $\overset{\circ}{\Gamma}{}^{A}_{BC}$ can be decomposed as (Schouten, 1954)

$$\overset{\circ}{\Gamma}{}^{A}_{BC} = \Gamma^{A}_{BC}[\mathbf{g}] + K_{BC}{}^{A}$$
$$K_{BC}{}^{A} = \tfrac{1}{2}g^{AD}(\tau_{BDC} + \tau_{CDB}) \tag{25}$$

where $\nabla^{g} = (\Gamma^{A}_{BC}[\mathbf{g}])$ denotes the Levi-Civita covariant derivative corresponding to the metric tensor $\mathbf{g}$. Using this connection decomposition, the geodesic equation (17), and (18), we can rewrite the condition (19) in the form (Bradfield, 1990)

$$Q_{ABC}v^{A}v^{B}v^{C} = 0 \tag{26}$$

It is easy to see that the tensor field Q_{CAB} defined by equations (22)–(24) fulfills the condition (26). Moreover,

$$Q_{CAB} = 0 \qquad \text{iff} \quad \tau_{ABC} = e_{CBA} \tag{27}$$

We see, that although Φ-geodesics are $\mathbf{g}$-geodesics only if the condition (27) is fulfilled, nevertheless the metric line element of the internal length measurement along a Φ-geodesic is equivalent to the affine line element of this geodesic [equation (20)]. Thus, an *affine transformation* of the teleparallel space $(\mathscr{B}, \nabla^{\Phi})$ that is a diffeomorphism of $\mathscr{B}$ preserving Φ-geodesics and their affine parameters preserves the property of lattice lines to be Φ-geodesics as well as [up to the affine transformation (20)] the internal length measurement along these lines. A local affine transformation (called also an infinitesimal affine motion) $X^{A} \to X^{A} + \epsilon v^{A}$ of a space with the covariant derivative $\nabla = (\Gamma^{A}_{BC})$ is defined by

$$\underset{\mathbf{v}}{\mathbf{L}}\,\Gamma^{A}_{BC} = 0 \tag{28}$$

where $\mathbf{L}$ denotes the Lie derivative operator. If $\nabla = \nabla^{\Phi}$, $\Phi = (\mathbf{E}_{a})$, the condition (28) is equivalent to [see equations (11) and (12)] (Yano, 1958)

$$\underset{\mathbf{v}}{\mathbf{L}}\,\underset{a}{e}^{A} = c^{b}{}_{a}\underset{b}{e}^{A}, \qquad c^{b}{}_{a} = \text{const} \tag{29}$$

If all the constants $c^{b}{}_{a}$ are zero, that is, if

$$\underset{\mathbf{v}}{\mathbf{L}}\,\underset{a}{e}^{A} = 0 \tag{30}$$

the local affine transformation is said to be *particular*. The maximal order of a group G_r of particular affine transformations is $r = n = 3$ ($n = 3$ is the dimension of the body manifold $\mathscr{B}$), and such a group exists iff [see (13)] (Yano, 1958)

$$\nabla^{\Phi}\boldsymbol{\tau}_{\Phi} = 0, \qquad \text{i.e.,} \qquad \tau^a{}_{bc} = \text{const} \tag{31}$$

Note that the maximal order of a group G_r of all affine transformations is $r = n(n+1) = 12$, and this group exists iff the space is Euclidean, the group being a general affine group (Yano, 1958). Namely, in this case we can take a rectilinear coordinate system (ξ^a) such that for $\mathbf{v} = v^a\partial_a$, $\partial_a = \partial/\partial\xi^a$, the condition (29) gives

$$v^a = c^a{}_b\xi^b + c^a \tag{32}$$

where $c^a = \text{const}$, which means that the particular local affine transformation is then a translation, and thus the general local particular affine transformation can be considered as a local *generalized translation* [in the teleparallel space $(\mathscr{B}, \nabla^{\Phi})$]. A teleparallelism fulfilling the condition (31) is said to be *closed*, and the corresponding distribution of dislocations is said to be *uniformly dense* (Trzęsowski, 1987). In this case the moving frame $\Phi = (\mathbf{E}_a)$ spans a three-dimensional real Lie algebra $\mathfrak{g}[\Phi]$ of Φ-parallel vector fields defined by (14). The Lie multiplication $[\,\cdot\,,\cdot\,]$ of this Lie algebra is defined by the commutation rules (3) with $C^c_{ab} = -\tau^c{}_{ab} = \text{const}$ [see equation (2b)]. Moreover, it follows from equation (6) that the Jacobi identity (e.g., Von Westenholz, 1978) of the Lie algebra $\mathfrak{g}[\Phi]$ reduces to the condition

$$\gamma^{ab}t_b = 0 \tag{33}$$

If the teleparallelism is closed, the Ricci tensor $R_{AB}[\mathbf{g}]$ of the Levi-Civita covariant derivative $\nabla^g = [\Gamma^A_{BC}[\mathbf{g}])$ can be computed explicitly in terms of the dislocation density tensor $\boldsymbol{\alpha}$ [equation (5)]. Namely, it follows from the decomposition (25) that the Ricci tensor $\ddot{R}_{AB}$ of the covariant derivative $\ddot{\nabla} = (\ddot{\Gamma}^A_{BC})$ has the form (Schouten, 1954)

$$\ddot{R}_{AB} = \ddot{R}_{CAB}{}^C = R_{AB}[\mathbf{g}] + \tfrac{1}{4}c_{AB} - \tfrac{1}{2}g^{CD}\nabla^g_C t_{ABD} - \tfrac{1}{2}T_{AB} \tag{34}$$

where $\ddot{R}_{ABC}{}^D$ are components of the curvature tensor of the covariant derivative $\ddot{\nabla}$, and we have denoted

$$\begin{aligned} c_{AB} &= \tau^D{}_{CA}\tau^C{}_{BD} \\ t_{ABD} &= 2\tau_{(AB)D} + t_{(B}g_{A)D} \\ T_{AB} &= \tau_{(AB)C}t^C + \tau^C{}_{D(B}\tau_{A)C}{}^D + \tfrac{1}{2}\tau_{AC}{}^D\tau_{BD}{}^C \\ \tau_{ABC} &= g_{AD}\tau^D{}_{BC}, \qquad \tau_{AB}{}^C = g^{CD}\tau_{ABD} \end{aligned} \tag{35}$$

If the teleparallelism is closed, then (Yano, 1958)

$$\overset{\circ}{R}_{AB} = \tfrac{1}{4} c_{AB} \tag{36}$$

and from equations (34)–(36) we obtain that

$$\begin{aligned} R_{AB}[\mathbf{g}] &= \nabla^{g}_{C} \Sigma_{AB}{}^{C} + \tfrac{1}{2}\sigma_{AB} \\ \Sigma_{AB}{}^{C} &= \gamma_{D(A} e_{B)}{}^{CD} + \tfrac{1}{2} g_{AB} t^{C} \\ \sigma_{AB} &= g_{AB} \gamma^{C}{}_{C} - \gamma^{C}{}_{A} \gamma_{CB} - \tfrac{1}{2} e_{CD(B} t^{C} \gamma^{D}{}_{A)} \end{aligned} \tag{37}$$

where [see equations (23) and (24)]

$$\begin{aligned} \gamma^{A}{}_{B} &= g_{BC} \gamma^{AC}, \qquad & \gamma_{AB} &= g_{AC} \gamma^{C}{}_{B} \\ \gamma^{C}{}_{C} &= g_{AB} \gamma^{AB}, \qquad & e_{A}{}^{BC} &= g_{AD} e^{DBC} \end{aligned} \tag{38}$$

3. UNIFORMLY DENSE DISTRIBUTIONS OF DISLOCATIONS

Let μ_Φ denote the Riemannian volume 3-form defined by the internal length measurement metric tensor $\mathbf{g}$ [see designations (24)]:

$$\mu_\Phi = \sqrt{g}\, dX^1 \wedge dX^2 \wedge dX^3 = E^1 \wedge E^2 \wedge E^3 \tag{39}$$

It can be shown that for a closed teleparallelism [equation (31)]

$$\overset{\circ}{\nabla} \boldsymbol{\tau}_\Phi = 0 \tag{40a}$$

$$\overset{\circ}{\nabla} \mu_\Phi = 0 \tag{40b}$$

where $\overset{\circ}{\nabla}$ is defined by equations (15) and (16), and thus [see (23)]

$$\overset{\circ}{\nabla} \boldsymbol{\gamma} = 0, \qquad \boldsymbol{\gamma} = \gamma^{AB} \partial_A \otimes \partial_B \tag{41a}$$

$$\overset{\circ}{\nabla} t = 0, \qquad t = t_A\, dX^A \tag{41b}$$

and [equation (33)]

$$\begin{aligned} \boldsymbol{\gamma} \mathbf{t} &= 0 \\ \mathbf{t} = t^a \mathbf{E}_a, &\qquad t^a = \delta^{ab} t_b \end{aligned} \tag{42}$$

where

$$\begin{aligned} \gamma^{ab} &= \overset{a}{e}_A \overset{b}{e}_B \gamma^{AB} = \gamma^{ba} = \text{const} \\ t_a &= \underset{a}{e}{}^{A} t_A = \text{const} \end{aligned} \tag{43}$$

From equation (41b) we obtain that

$$dt = 0 \tag{44}$$

i.e., at least locally, one has

$$t = d\varphi \tag{45}$$

where φ is a scalar. Let us consider the following representation of the tensor γ and the covector t (Trzęsowski, 1994, Section 3):

$$\begin{aligned} \gamma = \lambda^a \mathbf{e}_a \otimes \mathbf{e}_a, \qquad t = \mu e^3 \\ (\mathbf{e}_a, \mathbf{e}_b)_g = \delta_{ab}, \qquad \langle e^3, \mathbf{e}_a \rangle = \delta^3_a \end{aligned} \tag{46}$$

The conditions (33) and (43) mean that

$$\lambda^a = \text{const}, \qquad \mu = \text{const}, \qquad \lambda^3 \mu = 0 \tag{47}$$

If $\mathbf{t} \neq \mathbf{0}$ and $\gamma \neq \mathbf{0}$, then the conditions (42) and (47) mean that rank $\gamma = s$. $1 \leq s \leq 2$, and $\lambda^3 = 0$, $\mu \neq 0$. Moreover, it follows from the condition (41a) that then for every point $P \in \mathscr{B}$ there exists its coordinate neighborhood with coordinates $X = (X^A) = (x^\alpha, z^k)$, $1 \leq \alpha \leq s$, $s < k \leq 3$, such that the matrix $\|\gamma^{AB}(X)\|$ of γ components has the following form:

$$\|\gamma^{AB}(X)\| \overset{*}{=} \left\| \begin{matrix} \|\gamma^{\alpha\beta}(x)\| & 0 \\ 0 & 0 \end{matrix} \right\| \tag{48}$$

with $\|\gamma^{\alpha\beta}(x)\|$, $x = (x^\alpha)$, nondegenerate, and the number of coordinates x^α being independent of P (Rendal, 1992).

If rank $\gamma = 3$, then from condition (42) it follows that

$$\mathbf{t} = 0 \tag{49}$$

and the condition (41a) means that $\overset{\circ}{\nabla}$ is a (symmetric) metric connection. In this case [see equations (5)–(7)]

$$\tau^a{}_{bc} = e_{bcd}\gamma^{da}, \qquad \alpha^{ab} = \gamma^{ab} \tag{50}$$

It follows from the Bianchi classification of three-dimensional real Lie algebras (Barut and Rączka, 1977; Dubrovin *et al.*, 1979) that the only Lie algebras $\mathfrak{g}[\Phi]$ for which $\overset{\circ}{\nabla}$ is a metric connection are those of three-dimensional Euclidean rotations type ($\mathfrak{g}[\Phi] \cong so(3)$, sign $\gamma = (-, -, -)$) or three-dimensional Lorentz rotations type ($\mathfrak{g}[\Phi] \cong so(2, 1)$, sign $\gamma = (+, +, -)$). Let us introduce the metric tensor $\mathbf{g}_c$ by

$$\begin{aligned} \mathbf{g}_c = -\epsilon l_0^{-1} \gamma^{-1} = g_{(c)ab} E^a \otimes E^b \\ g_{(c)}{}^{ab} = -\epsilon l_0 \gamma^{ab}, \qquad g_{(c)}{}^{ab} g_{(c)bc} = \delta^a_c \\ [\mathbf{g}_c] = [l^2], \qquad [l_0] = [l] \end{aligned} \tag{51}$$

where l_0 is a constant, $[l] = \text{cm}$ in the cgs units system, $\epsilon = 1$ if $\mathfrak{g}[\Phi] \cong so(3)$, and $\epsilon = -1$ if $\mathfrak{g}[\Phi] \cong so(2, 1)$. For example, if the commutation rules (3) are defined by

$$[\mathbf{E}_1, \mathbf{E}_2] = \lambda \mathbf{E}_3, \qquad [\mathbf{E}_2, \mathbf{E}_3] = \epsilon\lambda \mathbf{E}_1, \qquad [\mathbf{E}_3, \mathbf{E}_1] = \epsilon\lambda \mathbf{E}_2$$
$$\lambda > 0, \qquad [\lambda] = [l^{-1}] \tag{52}$$

then

$$\gamma^{ab} = -\lambda\epsilon\eta^{ab}, \qquad l_0 = \lambda^{-1}$$
$$g_{(\epsilon)}{}^{ab} = \eta^{ab}, \qquad \|\eta^{ab}\| = \mathrm{diag}(1, 1, \epsilon) \tag{53}$$

It can be shown (Wolf, 1972) that if $\Phi = (\mathbf{E}_a)$ is a globally defined moving frame, $\mathbf{g}_\epsilon$ is a Riemannian ($\epsilon = 1$) or pseudo-Riemannian ($\epsilon = -1$) metric tensor on a connected manifold $\mathscr{B}$, and

$$(\mathbf{E}_a, \mathbf{E}_b)_{\mathbf{g}_\epsilon} = g_{(\epsilon)ab} = \mathrm{const} \tag{54}$$

then the following conditions are equivalent:

(a) $\overset{\circ}{\Gamma}{}^a_{bc} = \Gamma^a_{bc}[\mathbf{g}_\epsilon]$.
(b) $\mathbf{g}_\epsilon$-geodesics are Φ-geodesics with the same affine parameters.
(c) $\mathbf{E}_a$ are Killing vectors on $(\mathscr{B}, \mathbf{g}_\epsilon)$.

Here $\Gamma^a_{bc}[\mathbf{g}_\epsilon]$ are Christoffel symbols corresponding to $\mathbf{g}_\epsilon$. The conditions (54) and (c) mean that vector fields $\mathbf{E}_a$ are the so-called translations in $(\mathscr{B}, \mathbf{g}_\epsilon)$, i.e., infinitesimal isometries under which every point is moved over the same distance (Yano, 1958). If $\epsilon = 1$, then the condition (b) is equivalent to $Q_{CAB} = 0$ in equation (21) [cf. equation (27)] and generalizes the known theorem that each translation-invariant metric on R^n is consistent with the Euclidean parallelism (Wolf, 1972). Therefore, in this case (and only in this case), the property of an ideal (that is Euclidean) Bravais lattice that lattice lines are geodesics of the Euclidean parallelism is locally reconstructed in a continuized dislocated monocrystal. Both Lie algebras $\mathfrak{g}[\Phi]$ corresponding to the metric tensors $\mathbf{g}_\epsilon$, $\epsilon = 1$ or $\epsilon = -1$, describe *screw dislocations* (Trzęsowski, 1994, Section 3). Namely, $\mathfrak{g}[\Phi] \cong so(3)$ (corresponding to $\mathbf{g}_1$) describes a distribution of screw dislocations of the same type (right-handed or left-handed), while $\mathfrak{g}[\Phi] \cong so(2, 1)$ (corresponding to $\mathbf{g}_{-1}$) admits the existence of screw dislocations with opposite local Burgers vectors. These two types of continuous distributions of dislocations are called *orthogonal* (Euclidean or Lorentzian type) (Trzęsowski and Sławianowski, 1990). Note that the Lie algebra $\mathfrak{g}[\Phi] \cong so(3)$ may be also interpreted as describing a continuous counterpart of discrete disclinations (Trzęsowski, 1993). However, its "screw interpretation" seems more appropriate. This is because, in the continuized crystal approximation, disclinations are rather a type of a distribution of dislocations than a separate kind of line defect (Trzęsowski, 1993).

The case

$$\gamma = 0, \qquad t \neq 0 \tag{55}$$

describes a distribution of *edge dislocations* (and only edge) (Trzęsowski, 1994, Section 3). In this case

$$\begin{aligned} \tau^a{}_{bc} &= -t_{[b}\delta^a_{c]} \\ \alpha^{ab} &= \tfrac{1}{2}t_c e^{cab} = -\alpha^{ba} \end{aligned} \tag{56}$$

For example, if [cf. equations (46) and (47)]

$$t = \mu E^3, \qquad \text{i.e.,} \qquad t_a = \mu\delta^3_a \tag{57}$$

then the corresponding Lie algebra $\mathfrak{g}[\Phi]$ is defined by the following commutation rules:

$$\begin{gathered} [\mathbf{E}_1, \mathbf{E}_2] = 0, \qquad [\mathbf{E}_3, \mathbf{E}_2] = \kappa\mathbf{E}_2, \qquad [\mathbf{E}_3, \mathbf{E}_1] = \kappa\mathbf{E}_1 \\ \kappa = \mu/2 > 0 \end{gathered} \tag{58}$$

It follows from (37) that, in the case (55), the Riemannian material space $(\mathscr{B}, \mathbf{g})$ has a constant scalar curvature K, $[K] = [l^{-2}]$, and

$$R_{AB}[\mathbf{g}] = \alpha g_{AB} \tag{59a}$$

$$\alpha = \tfrac{1}{2}\nabla^g_A t^A = 2K = \text{const} \tag{59b}$$

If the condition (49) is fulfilled, then

$$R_{AB}[\mathbf{g}] = \nabla^g_C \gamma_{D(A} e_{B)}{}^{CD} + \tfrac{1}{2}(\gamma_{AB}\gamma^C{}_C - \gamma^C{}_A\gamma_{CB}) \tag{60}$$

and the scalar curvature K is given by

$$\begin{aligned} K &= \tfrac{1}{6}R \\ R &= g^{AB}R_{AB}[\mathbf{g}] = \tfrac{1}{2}[(\gamma^C{}_C)^2 - \gamma^{AB}\gamma_{AB}] \end{aligned} \tag{61}$$

It follows from equations (46), (47), and (61) that

$$R = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3 = \text{const} \tag{62}$$

In particular, if rank $\gamma = 3$ and the tensor γ is defined by (53), then

$$K = \frac{1 + 2\epsilon}{6}\lambda^2, \qquad [\lambda] = [l^{-1}] \tag{63}$$

and for $\epsilon = 1$

$$R_{AB}[\mathbf{g}] = \lambda^2 g_{AB}, \qquad K = \tfrac{1}{2}\lambda^2 \tag{64}$$

However, although the scalar curvature K defined by equations (61) and (62) is a constant, the Ricci tensor cannot be in general reduced to the form

(59a), and ($\mathscr{B}$, **g**) is not a flat space even in the case $K = 0$. For example, if g[Φ] is isomorphic with the so-called Weyl Lie algebra defined by

$$[\mathbf{E}_1, \mathbf{E}_2] = 0, \qquad [\mathbf{E}_2, \mathbf{E}_3] = \lambda \mathbf{E}_1, \qquad [\mathbf{E}_3, \mathbf{E}_1] = 0 \tag{65}$$

then $\|\gamma^{ab}\| = \lambda\, \mathrm{diag}(-1, 0, 0)$, $t = 0$, and from equation (62) we obtain that $R = 0$, but ($\mathscr{B}$, **g**) is not a flat space.

4. SLIP SURFACES

Let us consider a *local Burgers vector* **b** defined by (Trzęsowski, 1994, Section 3)

$$\begin{aligned} \rho \mathbf{b} = \mathbf{l}\boldsymbol{\alpha}, \qquad &\text{i.e.,} \qquad \rho b^A = l_B \alpha^{BA} \\ \mathbf{l} = l^A \partial_A, \qquad &l_A = g_{AB} l^B, \qquad l^A l_A = 1 \end{aligned} \tag{66}$$

where **l** denotes the unit vector field tangent to a dislocation line, and ρ, $[\rho] = [l^{-2}]$, is the scalar density of dislocations (indendent of the choice of **l**). The plane $\pi(\mathbf{l}, \mathbf{b})$ containing vectors **l** and **b** is interpreted as a *local slip plane*. If the teleparallelism is closed, then it follows from equations (5)–(7) and (42) that

$$b^A t_A = 0 \tag{67}$$

and thus, if additionally

$$l^A t_A = 0 \tag{68}$$

the vector field **t** is then normal to the local slip plane at each point of the body. The condition (45) means then that surfaces $\varphi = \mathrm{const}$ are the so-called *slip surfaces*. For example, it follows from equations (45) and (59) that for the uniformly dense distribution of *edge dislocations* defined by (55), slip surfaces are defined by

$$\begin{aligned} \Delta_g \varphi &= 4K \\ K = \mathrm{const}, \qquad &[K] = [l^{-2}] \end{aligned} \tag{69}$$

where Δ_g is the Laplace–Beltrami operator on ($\mathscr{B}$, **g**):

$$\Delta_g \varphi = g^{AB} \nabla_A^g \nabla_B^g \varphi = g^{-1/2} \partial_A (g^{1/2} g^{AB} \partial_B \varphi) \tag{70}$$

Note that from equations (23) and (66) it follows that in the case (55)

$$\rho b^A = \tfrac{1}{2} t_C l_B e^{CBA} \tag{71}$$

Thus, the condition (67) is then an identity and, independently of the choice of the dislocation line fulfilling the condition (68), the following relation is valid:

$$\rho b = \mu/2 = \text{const}$$
$$b^2 = b_A b^A, \qquad \mu^2 = t_A t^A, \qquad [b] = [l], \qquad [\mu] = [l^{-1}] \tag{72}$$

The slip direction is necessarily always parallel to the Burgers vector of the dislocation responsible for slip (Hull and Bacon, 1984). This suggests that a local generalized translation [in the teleparallel space $(\mathscr{B}, \nabla^\Phi)$—see the commentary following equation (32)], parallel to the local Burgers vector **b** may be considered as the example of a *local slip* in the continuized dislocated crystal. Note that such local slips preserve the property of lattice lines (in a continuized dislocated monocrystal—Section 1) to be Φ-geodesics as well as preserve [up to the affine transformation (20)] an internal length measurement along these lines (Section 2). Since (Yano, 1958)

$$\underset{\mathbf{b}}{\mathrm{L}} \underset{a}{e}{}^A = -\underset{a}{e}{}^B b_B^A$$
$$b_A^B = \nabla_A^\Phi b^B + b^C \tau_{CA}{}^B, \qquad \tau_{AB}{}^C = g^{CD} g_{AE} \tau^E{}_{BD} \tag{73}$$

we obtain, taking into account equations (23b), (29)–(31), (45), (66), (67), and (73), that

$$\nabla_A^\Phi b^B = e_{ACD} b^C \gamma^{DB} - \tfrac{1}{2} t_A b^B$$
$$t^A b_A = 0, \qquad t_A = \partial_A \varphi \tag{74}$$

The dislocation density tensor $\boldsymbol{\alpha}$ of a uniformly dense distribution of dislocations is a Φ-parallel tensor field [i.e., $\alpha^{ab} = \text{const}$ in equation (5)]. Therefore, the vector field $\mathbf{v} = \rho\mathbf{b}$ defined by (66) with **l** being a Φ-parallel vector field should also be Φ-parallel, and then [see equation (14a)]

$$\nabla_A^\Phi b^B = -\tfrac{1}{2}(\partial_A \zeta) b^B$$
$$\zeta = 2\ln(\rho/\rho_0), \qquad \nabla^\Phi \mathbf{l} = 0 \tag{75}$$

Finally, in this case, we obtain from equations (14b), (74), and (75) that slip surfaces $\varphi = \text{const}$ defined by the conditions

$$\partial_A(\varphi - \zeta) = 2 e_{ACD} \beta^C \gamma^{DB} \beta_B$$
$$\beta^A \partial_A \varphi = 0, \qquad \rho b = \kappa = \text{const} \tag{76}$$
$$\beta^A = b^A/b, \qquad \beta_A = g_{AB}\beta^B, \qquad b^2 = g_{AB} b^A b^B$$

admit local slips of the considered type. In particular, for the distribution of edge dislocations defined by (55), these slip surfaces are defined by

$$\varphi = 2\ln(\rho/\rho_0) \tag{77}$$

Thus, in this case, we can compute the scalar density of dislocations ρ from equations (69) and (77) and the modulus b of the local Burgers vector from (72).

5. FINAL REMARKS

It follows from equations (41) and (42) that, for a uniformly dense distribution of dislocations with rank $\gamma = 2$, the pair (γ, t) can be treated as defining a Galilei-like structure in the Riemannian space $(\mathscr{B}, g)$ that is endowed with the covariant derivative $\overset{\circ}{\nabla} = (\overset{\circ}{\Gamma}{}^A_{BC})$ [see (25)] as the so-called symmetric Galilei covariant derivative (e.g., Künzle, 1972). The maximal integral manifolds of this Galilei-like structure have the form $\varphi = \text{const}$ [equation (45)]. The general symmetric Galilei-covariant derivative is defined by the conditions (41) and (42) with a (symmetric) covariant derivative $\nabla = (\Gamma^A_{BC})$ instead of $\overset{\circ}{\nabla}$, and has the following form (Künzle, 1972):

$$\begin{aligned} \Gamma^A_{BC} &= \overset{\circ}{\Gamma}{}^A_{BC} + \gamma^{AD}\chi_{D(B}t_{C)} \\ \gamma^{AB}t_B = 0, \qquad &\operatorname{rank}\gamma = 2, \qquad [\chi_{AB}] = [l] \end{aligned} \tag{78}$$

where $\chi = \frac{1}{2}\chi_{AB}\,dX^A \wedge dX^B$, $\chi_{AB} = -\chi_{BA}$, is a 2-form, and the condition (44) is fulfilled. The covariant derivative ∇ defined by (78) admits μ_Φ [equation (39)] as an invariant volume 3-form [i.e., the condition (40b) with ∇ instead of $\overset{\circ}{\nabla}$ is fulfilled].

The conditions (17), (21), and (26) for agreement between an internal length measurement and affine intervals along $\overset{\circ}{\nabla}$-geodesics are valid for all symmetric covariant derivatives (Bradfield, 1990). In particular, for the Galilei-covariant derivative ∇ defined by (78), we have

$$\begin{aligned} \nabla_C g_{AB} &= Q_{CAB} - M_{CAB} \\ M_{CAB} &= 2N_{C(AB)} \\ N^A_{BC} = \gamma^{AD}\chi_{D(B}t_{C)}, &\qquad N_{CAB} = g_{BD}N^D_{CA} \end{aligned} \tag{79}$$

where Q_{CAB} is given by equations (22) and (23), and thus these conditions reduce then to the condition [cf. equation (26)]

$$(Q_{CAB} - M_{CAB})v^C v^A v^B = -M_{CAB}v^C v^A v^B = 0 \tag{80}$$

which is fulfilled for the vector field v^A tangent to a ∇-geodesic [equation (17) with Γ^A_{BC} instead of $\overset{\circ}{\Gamma}{}^A_{BC}$] iff any of the following conditions is fulfilled:

$$\gamma^{AB}v_B = 0 \tag{81a}$$

$$\chi_{AB}v^B = 0 \tag{81b}$$

$$t_A v^A = 0 \tag{81c}$$

where $v_A = g_{AB}v^B$. Any of these conditions ensures the existence of a distinguished *consistency direction* between an internal length measurement and the Galilei-like structure. For example, if [cf. equations (18) and (20)]

$$\begin{aligned} &t_A = \mu n_A, \qquad v^A = a n^A, \qquad n_A n^A = 1 \\ &[n_A] = [n^A] = [a] = [1], \qquad [\mu] = [l^{-1}] \end{aligned} \tag{82}$$

where μ and a are positive constants, then the unit vector field n^A defines, according to equations (33) and (81a), the consistency direction and can be considered as a counterpart of the so-called timelike vector field. Moreover, a coordinate system in which the representation (48) (with $s = 2$) of the tensor field γ holds and $n_A = \delta^3_A$ defines the so-called adapted coordinates (e.g., Künzle, 1972).

We see that the geometry associated with a uniformly dense distribution of dislocations (with rank $\gamma = 2$) defines a foliation of the material space $(\mathscr{B}, \mathbf{g})$ consisting of slip surfaces in like manner as a Galilei structure defines a foliation of the space-time consisting of its spacelike subspaces. Note that a slip plane normal to the $\mathbf{t}$ direction [equations (66)–(68)] can be considered as a *local glide plane*, that is, a local slip plane in which a (local) translational motion (called the glide motion) of a dislocation occurs (Hull and Bacon, 1984). In this case, surfaces of the Galilei-like foliation are those that admit a glide motion of dislocations [see also the commentary preceding (73)]. Therefore, the analogy with a Galilei structure of the space-time has a geometrical as well as a dynamical meaning. The consequences of this analogy will be studied elsewhere.

REFERENCES

Barut, A. O., and Rączka, R. (1977). *Theory of Group Representations and Applications*, PWN (Polish Scientific Publishers), Warsaw [in Polish].

Bilby, B. B., Bullough, R., Gardner, L. A., and Smith, E. (1958). *Proceedings of the Royal Society*, **244**, 588.

Bradfield, T. (1990). *General Relativity and Gravitation*, **22**(6), 665.

Dubrovin, B. A., Novikow, S. P., and Fommienko, A. T. (1979). *Modern Geometry*, Nauka, Moscow [in Russian].

Hull, D., and Bacon, D. J. (1984). *Introduction to Dislocations*, Pergamon Press, Oxford.

Kröner, E. (1984). In *Dislocations in Solids, Some Recent Advances*, K. Markensoff, ed., American Society of Mechanical Engineers, New York.

Kröner, E. (1985). In *Dislocations and Properties of Real Materials*, Institute of Metals, London, p. 67.

Kröner, E. (1986). *Zeitschrift für Angewandte Mathematik und Mechanik*, **66**, T284.

Künzle, H. P. (1972). *Annales de l'Institut Henri Poincaré*, **XVII**(4), 337.

Rendall, A. D. (1992). *Proceedings of the Royal Society: Mathematical and Physical Sciences*, **438**, 341.

Schouten, J. A. (1954). *Ricci-Calculus*, Springer-Verlag, Berlin.

Trzęsowski, A. (1987). *International Journal of Theoretical Physics*, **26**(11), 1059.

Trzęsowski, A. (1993). *Reports on Mathematical Physics*, **32**(1), 71.

Trzęsowski, A. (1994). *International Journal of Theoretical Physics*, this issue.

Trzęsowski, A., and Sławianowski, J. J. (1990). *International Journal of Theoretical Physics*, **29**(11), 1239.

Von Westenholz, C. (1978). *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam.

Wolf, I. A. (1972). *Journal of Differential Geometry*, **6**, 317.

Yano, K. (1958). *The Theory of Lie Derivatives and its Applications*, North-Holland, Amsterdam.